Basic Statistics

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Contents

0 Revision History

P1708R1

- An accumulator object is proposed to allow for the computation of statistics in a **single** pass over a sequence of values.

P1708R2

- Reformatted using LaTeX.
- A (possible) return to freestanding functions is proposed following discussions of the accumulator object of the previous version.

P1708R3

- **Geometric mean** is proposed, since it exists in Calc, Excel, Julia, MATLAB, Python, R and Rust.
- **Harmonic mean** is proposed, since it exists in Calc, Excel, Julia, MATLAB, PHP, Python, R and Rust.
- **Weighted means**, **median**, **mode**, **variances** and **standard deviations** are proposed, since they exist (with the exception of mode) in MATLAB and R.
- **Quantile** is proposed, since it is more generic than median and exists in Calc (percentile), Excel (percentile), Julia, MATLAB, PHP (percentile), R and SQL (percentile).
- **Skewness** is proposed, since it exists in Calc, Excel, Julia, MATLAB, PHP, R, Rust, SAS and SQL and was recommended as part of a presentation to SAS corporation.
- **Kurtosis** is proposed, since it exists in Calc, Excel, Julia, MATLAB, PHP, R, Rust, SAS and SQL and was recommended as part of a presentation to SAS corporation.
- Both **freestanding functions** and **accumulator objects** are proposed, since they (largely) have distinct purposes.
- **Iterator pairs** are replaced by **ranges**, since ranges simplify predicates (as comparisons and projections).

P1708R4

- Parameter `data_t` (corresponding to values `population_t` and `sample_t`) of **variance** and **standard deviation** are replaced by **delta degrees of freedom**, since this is done in Python (NumPy).
- In the case of a **quantile** (or median), specific methods of interpolation between adjacent values is proposed, since this is done in Python (NumPy).
- `stats_error`, previously a constant, is replaced by a **class**.

P1708R5

- **Quantile** (and **median**) and **mode** are deferred to a future proposal, given ongoing unresolved issues relating to these statistics.
- `stats_error`, an **exception**, is removed, since (C++) math funtions do not throw exceptions.
- The ability to create **custom** accumulator objects is proposed, since this is done in Boost Accumulators.
- `stats_result_t` is introduced so as to simplify (function) signatures.
- Various errors in statistical formulas are corrected.
- Various functions, objects (classes) and parameters are renamed so as to be more meaningful.
- Various technical errors relating to ranges and execution policy are corrected.

P1708R6

- `stats_result_t` is removed, since return type is deduced from projection.
- **Accumulator** objects are revised so as to be simpler and allow for parallel implementations.
- `stat_accum` and `weighted_stat_accum` are removed, since they are no longer needed.
- **Concepts** are removed so as to allow for **custom** data types.
- **Projections** are removed, since **views** already offer such functionality.
- Numerous functions and classes are renamed so as to be more meaningful.
- Reformatted so as to fulfill the specification style guidelines and **standardese**.

P1708R7

- **Unweighted** and **weighted** functions are combined so as to take advantage of **overloading**.
- The presentation of formulas is simplified.
- **Derivations** of skewness and kurtosis formulas are given.
- The wording of the technical specifications is updated.
- Further reformatted so as to fulfill the specification style guidelines and **standardese**.

P1708R8

- Statistics are reordered as first, second, third and fourth moments.
- **Constructors** are no longer **`explicit`**.
- `value` member function of accumulator objects is simplified.
- The name `stats` is replaced by the more meaningful name `statistics`.

1 Introduction

This document proposes an extension to the C++ library, to support **basic statistics**.

2 Motivation and Scope

Basic statistics, **not** presently found in the standard (including the special math library), frequently arise in **scientific** and **industrial**, as well as **general**, applications. These functions do exist in Python [1], the foremost competitor to C++ in the area of **machine learning**, along with Calc [2], Excel [3], Julia [4], MATLAB [5], PHP [6], R [7], Rust [8], SAS [9], SPSS [10] and SQL [11]. Further need for such functions has been identified as part of **SG19** (machine learning) [12].

This is not the first proposal to move statistics in C++. In 2004, a number of statistical distributions were proposed in [13]. Additional distributions followed in 2006 [14]. Statistical distributions ultimately appeared in the C++11 standard [15]. Distributions, along with statistical tests, are also found in Boost [16]. A series of special mathematical functions later followed as part of the C++17 standard [17]. A C library, GNU Scientific Library [18], further includes support for statistics, special functions and histograms.

Five statistics are defined in this proposal. Two (univariate) statistics, specifically **percentile** (and **median**) and **mode**, are **not** included in this proposal. These more involved statistics are deferred to a **future** proposal. Like existing entities of the (C++) standard library, this proposal specifies only the interface of functions and objects, meaning that a variety of implementations are possible. This enables a vendor to favor accuracy [19] over performance for instance.

2.1 Mean

The *arithmetic mean* [20, 21] of the values $x_1, x_2, \ldots, x_n$ ($n \geq 1$), denoted μ or $\bar{x}$ in the case of a **population** [20] or **sample** [20], respectively, is defined as

$$\frac{1}{n}\sum_{i=1}^{n} x_i. \tag{1}$$

The *weighted* arithmetic mean [22, 21], for weights $w_1, w_2, \ldots, w_n$, denoted μ^* or $\bar{x}^*$ in the case of a **population** or **sample**, respectively, is defined as

$$\frac{1}{V_1}\sum_{i=1}^{n} w_i x_i, \tag{2}$$

where $V_1 = \sum_{i=1}^{n} w_i$. The *geometric* mean [20] is defined as

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \tag{3}$$

and the **weighted** geometric mean [22] is defined as

$$\left(\prod_{i=1}^{n} x_i^{w_i}\right)^{V_1^{-1}}. \tag{4}$$

The *harmonic* mean [20] of the positive values $x_i > 0$ is defined as

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} \tag{5}$$

and the **weighted** harmonic mean [23] is defined as

$$\frac{\sum\limits_{i=1}^{n} w_i}{\sum\limits_{i=1}^{n} \frac{w_i}{x_i}}. \tag{6}$$

Each of the arithmetic, geometric and harmonic means can be (accurately) computed in **linear** time [24]. When computing the associated sums of these means, and indeed any sum in this proposal, robust methods [25, 26] ought to be considered.

2.2 Variance

The **population** *variance* [20, 21] of the values ($n \geq 1$), denoted σ^2, is defined as

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 \tag{7}$$

and the **sample** variance [20, 21] ($n \geq 2$), denoted s^2, is defined as

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2. \tag{8}$$

The **weighted population** variance [21, 27], denoted $\hat{\sigma}^2$, is defined as

$$\frac{1}{V_1}\sum_{i=1}^{n} w_i (x_i - \mu^*)^2 \tag{9}$$

and the **weighted sample** variance [21, 27], also used in the case of *reliability weights* [28], denoted $\hat{s}^2$, is defined as

$$\frac{V_1}{V_1^2 - V_2}\sum_{i=1}^{n} w_i (x_i - \bar{x}^*)^2, \tag{10}$$

where $V_2 = \sum_{i=1}^{n} w_i^2$. Population and sample variance (and standard deviation) are computed using factors of $1/n$ and $1/(n-1)$, respectively. Other factors might be used instead, $1/(n-1.5)$ as an example [29, 30]. To allow for such factors, this proposal, like NumPy [31], enables one to specify *delta degrees of freedom* [31], a value subtracted from n. Variance (and standard deviation) can be computed in **linear** time [24, 32, 33].

2.3 Standard Deviation

The **population** *standard deviation* [20] of the values ($n \geq 1$), denoted σ, is defined as the square root of the population variance. The **sample** standard deviation [20] ($n \geq 2$), denoted s, is defined as the square root of the sample variance. Likewise, the **weighted population** standard deviation, denoted $\hat{\sigma}$, is defined as the square root of the weighted population variance and the **weighted sample** standard deviation [34], denoted $\hat{s}$, is defined as the square root of the weighted sample variance.

2.4 Skewness

The **population** *skewness* [20, 21], a measure of the **symmetry** [20] of the values ($n \geq 3$), is defined as

$$\frac{1}{n\sigma^3} \sum_{i=1}^{n} \left(x_i - \mu\right)^3 \tag{11}$$

and the **sample** skewness [20, 21] is defined as

$$\frac{n}{\left(n-1\right)\left(n-2\right)s^3} \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^3 . \tag{12}$$

The **weighted population** skewness [21] is defined as

$$\frac{1}{V_1 \hat{\sigma}^3} \sum_{i=1}^{n} w_i \left(x_i - \mu^*\right)^3 \tag{13}$$

and the **weighted sample** skewness [21] is defined as

$$\frac{\left(V_1^2 - V_2\right)^{3/2}}{V_1 \left(V_1^3 - 3V_1V_2 + 2V_3\right) \hat{\sigma}^3} \sum_{i=1}^{n} w_i (x_i - \bar{x}^*)^3, \tag{14}$$

where $V_3 = \sum_{i=1}^{n} w_i^3$. Skewness (and kurtosis) can be computed in **linear** time [24, 35]. When scaled by the factor

$$\frac{\sqrt{n\left(n-1\right)}}{n-2}, \tag{15}$$

skewness becomes the *adjusted Fisher-Pearson standardized moment coefficient* [36]. Derivations of Equations (11), (12), (13) and (14) are given in Appendix B.1.

2.5 Kurtosis

The *Pearson* [37] (non-excess) **population** *kurtosis* [21, 38, 39], a measure of the "**tailedness**" [39] of the values ($n \geq 4$), is defined as

$$\frac{1}{n\sigma^4} \sum_{i=1}^{n} \left(x_i - \mu\right)^4 \tag{16}$$

and the Pearson **sample** kurtosis [21] is defined as

$$\frac{n^2 - 2n + 3}{\left(n-1\right)\left(n-2\right)\left(n-3\right)s^4} \sum_{i=1}^{n} (x_i - \bar{x})^4 - \frac{3n\left(2n-3\right)\sigma^4}{\left(n-1\right)\left(n-2\right)\left(n-3\right)s^4}. \tag{17}$$

The **weighted** Pearson **population** kurtosis [21] is defined as

$$\frac{1}{V_1 \hat{\sigma}^4} \sum_{i=1}^{n} w_i \left(x_i - \mu^*\right)^4 \tag{18}$$

and the **weighted** Pearson **sample** kurtosis [21] is defined as

$$\frac{\left(V_1^2 - V_2\right)\left(V_1^4 - 3V_1^2V_2 + 2V_1V_3 + 3V_2^2 - 3V_4\right)}{V_1^3 \left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right) \hat{\sigma}^4} \sum_{i=1}^{n} w_i (x_i - \bar{x}^*)^4 - \frac{3\left(V_1^2 - V_2\right)\left(2V_1^2V_2 - 2V_1V_3 - 3V_2^2 + 3V_4\right)}{V_1^2 \left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}, \tag{19}$$

where $V_4 = \sum_{i=1}^{n} w_i^4$. The *Fisher* [37] or *excess* [38] **population** kurtosis is defined as

$$\frac{1}{n\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^4-3 \tag{20}$$

and the excess **sample** kurtosis [21] is defined as

$$\frac{n(n+1)}{(n-1)(n-2)(n-3)s^4}\sum_{i=1}^{n}(x_i-\bar{x})^4-\frac{3n^2\sigma^4}{(n-2)(n-3)s^4}. \tag{21}$$

The **weighted** excess **population** kurtosis [21] is defined as

$$\frac{1}{V_1\hat{\sigma}^4}\sum_{i=1}^{n}w_i(x_i-\mu^*)^4-3 \tag{22}$$

and the **weighted** excess **sample** kurtosis [21] is defined as

$$\frac{\left(V_1^2-V_2\right)\left(V_1^4-4V_1V_3+3V_2^2\right)}{V_1^3\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)\hat{\sigma}^4}\sum_{i=1}^{n}w_i(x_i-\bar{x}^*)^4-\frac{3\left(V_1^2-V_2\right)\left(V_1^4-2V_1^2V_2+4V_1V_3-3V_2^2\right)}{V_1^2\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)}. \tag{23}$$

Derivations of Equations (16), (17), (18), (19), (20), (21), (22) and (23) are given in Appendix B.2.

3 Impact on the Standard

This proposal is a pure **library** extension.

4 Design Decisions

The discussions of the following sections address the concerns that have been raised in regards to this proposal.

4.1 Freestanding Functions vs. Accumulator Objects

Perhaps the most significant concern stemming from this proposal is that of (freestanding) functions versus (accumulator) objects. In the first incarnation of this proposal, namely P1708R0, functions were (exclusively) proposed. **Functions** are useful when one wishes to compute a **single** statistic. Objects were introduced in P1708R1 and P1708R2, which allow a user to (efficiently) compute **more than one** statistic in a **single** pass over the values, an idea borrowed from Boost Accumulators [40]. Like Boost Accumulators, a programmer has the ability to create **custom** objects. Given that each of these two paradigms has merit, with functions again being most useful in the case of the computation of a single statistic and objects being more attractive in instances in which multiple statistics are computed, the decision has been made to incorporate **both** such models into this proposal. Users are thus able to choose the approach that best fits with their design rather than being forced to use one of two paradigms. Note that objects are declared prior to functions in Section 5, as some vendors might wish to implement functions using objects.

4.2 Overloaded Accumulator Objects

It has been suggested that unweighted and weighted variants of an object be combined into a single object. This combined object would have two (overloaded) functions `operator()`. This might be feasible in some cases, mean for instance. Such an object could take the form

```
template<class T, class W = T>
class mean_accumulator
{
public:
  constexpr mean_accumulator() noexcept;
  constexpr void operator()(const T& x);             // unweighted variant
  constexpr void operator()(const T& x, const W& w); // weighted variant
  constexpr void unweighted_finalize();              // unweighted variant
  constexpr void weighted_finalize();                // weighted variant
  constexpr T value();
};
```

Note the need for (two) functions `unweighted_finalize` and `weighted_finalize` to perform the correct post-processing following accumulation, functionality that is presently merged into the (single) function `value`. Strictly unweighted or weighted **custom** objects would not require two functions **`operator`**`()` and two post-processing functions, another potential point of confusion. Moving on, combined objects are less intuitive in the case of variance and standard deviation, in which population and sample variants exist, namely

```
template<class T, class W = T>
class variance_accumulator
{
public:
  // unweighted variant uses ddof
  constexpr variance_accumulator(T ddof) noexcept;

  // weighted variant uses enumerated value
  constexpr variance_accumulator(std::statistics_data_kind dkind) noexcept;

  constexpr void operator()(const T& x);              // unweighted variant
  constexpr void operator()(const T& x, const W& w); // weighted variant
  constexpr void unweighted_finalize();               // unweighted variant
  constexpr void weighted_finalize();                 // weighted variant
  constexpr T value();
};
```

Delta degrees of freedom (`ddof`) distinguish population and sample variants in the unweighted case, whereas enumerated values are used in the weighted case. As a result, these objects require multiple constructors, of which the correct one must be invoked in order to compute the desired statistic.

4.3 Trimmed Mean

The issue of a trimmed mean is raised in [41]. A $p\%$ *trimmed* mean [42] is one in which each of the $(p/2)\%$ **highest** and **lowest** values (of a **sorted** range) are excluded from the computation of that mean. This feature would require that the values of a given range either be **presorted** or **sorted** as part of the computation of a mean. As an author, Phillip Ratzloff feels (a sentiment that was echoed by the author of [41]) that one might handle this (and other similar) matter via **ranges**, specifically by using a statement of the form

```
auto m = data | std::ranges::sort | trim(p) | std::mean;
```

4.4 Special Values

Much like the question of the trimmed mean of the previous section, special values, such as $\pm\infty$ and **NaN**, are readily addressed using **ranges**, a motivating factor for the introduction of ranges into this proposal. As a result, a programmer might handle such values using, as an example, a statement of the form

```
auto m = data | std::ranges::filter([](auto x) { return !std::isnan(x); }) | std::mean;
```

4.5 Projections

The functions and objects of P1708R3, P1708R4 and P1708R5 employ projections as a means of accessing individual components of aggregate entities. Given that such functionality is available through the use of **views**, projections have been removed, thereby yielding simpler functions and objects. This is much like the approach suggested in Sections 4.3 and 4.4. An example that demonstrates the use of views is presented in Appendix A.

4.6 Concepts

Much like `std::complex`, the proposed (template) functions and objects are defined for each of the (C++) **arithmetic** types, **except** for **`bool`**. Also like `std::complex`, the effect of instantiating the templates for any other type is unspecified. A programmer can therefore attempt to use **custom** types with the proposed functions and objects. It is felt that the added flexibility afforded by not using **concepts** to strictly limit functions and objects to arithmetic types is in the interest of the C++ community. In fact, several concerned parties reached out to the authors of this proposal in regards to this matter, all of whom suggested that this flexible approach be taken. Note that concepts are still employed in the case of **execution policy**, namely `std::is_execution_policy_v`, in which a fixed set of policies exists.

4.7 Header and Namespace

Early versions of this proposal, specifically P1708R0, P1708R1 and P17082, request that the proposed functions and objects be placed into the `<numeric>` header. Since P1708R3, it has instead been suggested that the function and objects be placed into a (new) **header** `<statistics>`, just as was done with the rational arithmetic of `<ratio>`, probability distributions of `<random>`, bit operations of `<bit>` and constants of `<numbers>`. Like rational arithmetic, probability distributions, bit operations and constants, basic statistics fit into the existing `std` **namespace**.

4.8 `std::expected`

The prospect of returning an `std::expected` object from the `value` member function of accumulator objects has been raised. Specifically, it has been suggested to do so in the case that there are too few elements in a range to which a particular accumulator is applied. Presently, `std::expected` objects are not used (elsewhere) in this proposal, such as in the case of a NaN, nor among the mathematical functions of the C++ library. It would therefore be asymmetric to have some situations return an `std::expected` object but not others.

5 Technical Specifications

The templates of the classes and functions specified in this section are defined for each of the arithmetic types, except for **`bool`** . The effect of instantiating the templates for any other type is unspecified. Parallel function overloads follow the requirements of [algorithms.parallel].

5.1 Header `<statistics>` synopsis [statistics.syn]

```
#include <execution>

namespace std {

// data types

enum class statistics_data_kind { population, sample };

enum class statistics_skewness_kind { adjusted, unadjusted };

enum class statistics_kurtosis_kind { excess, nonexcess };

// accumulator objects

template<class T>
class mean_accumulator;

template<class T, class W = T>
class weighted_mean_accumulator;

template<class T>
class geometric_mean_accumulator;

template<class T, class W = T>
class weighted_geometric_mean_accumulator;

template<class T>
class harmonic_mean_accumulator;

template<class T, class W = T>
class weighted_harmonic_mean_accumulator;

template<class T>
class variance_accumulator;
```

```cpp
template<class T, class W = T>
class weighted_variance_accumulator;

template<class T>
class standard_deviation_accumulator;

template<class T, class W = T>
class weighted_standard_deviation_accumulator;

template<class T>
class skewness_accumulator;

template<class T, class W = T>
class weighted_skewness_accumulator;

template<class T>
class kurtosis_accumulator;

template<class T, class W = T>
class weighted_kurtosis_accumulator;

// accumulator objects accumulation functions

template<ranges::input_range R, class ...Accumulators>
constexpr void statistics_accumulate(R&& r, Accumulators&&... acc);

template<ranges::input_range R, ranges::input_range W, class ...Accumulators>
constexpr void statistics_accumulate(R&& r, W&& w, Accumulators&&... acc);

template<class ExecutionPolicy, ranges::input_range R, class ...Accumulators>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
void statistics_accumulate(ExecutionPolicy&& policy, R&& r, Accumulators&&... acc);

template<class ExecutionPolicy,
  ranges::input_range R, ranges::input_range W,
  class ...Accumulators>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
void statistics_accumulate(ExecutionPolicy&& policy, R&& r, W&& w, Accumulators&&... acc);

// freestanding functions

template<ranges::input_range R>
constexpr auto mean(R&& r) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto mean(ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto geometric_mean(R&& r) -> ranges::iterator_t<R>::value_type;
```

```
template<ranges::input_range R, ranges::input_range W>
constexpr auto geometric_mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto geometric_mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto geometric_mean(
  ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto harmonic_mean(R&& r) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto harmonic_mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto harmonic_mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto harmonic_mean(
  ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto variance(R&& r, typename ranges::iterator_t<R>::value_type ddof) ->
  ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto variance(R&& r, W&& w, std::statistics_data_kind dkind) ->
  ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto variance(
  ExecutionPolicy&& policy,
  R&& r,
  typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto variance(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  std::statistics_data_kind dkind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto standard_deviation(
  R&& r, typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
```

```
constexpr auto standard_deviation(R&& r, W&& w, std::statistics_data_kind dkind) ->
  ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto standard_deviation(
  ExecutionPolicy&& policy,
  R&& r,
  typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto standard_deviation(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  std::statistics_data_kind dkind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto skewness(
  R&& r, statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto skewness(
  R&& r, W&& w, statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto skewness(
  ExecutionPolicy&& policy,
  R&& r,
  statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto skewness(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R>
constexpr auto kurtosis(
  R&& r, statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto kurtosis(
  R&& r, W&& w, statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto kurtosis(
```

```
  ExecutionPolicy&& policy,
  R&& r,
  statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto kurtosis(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;
}
```

5.2 Accumulator Objects

The accumulator objects specified in this section are trivially copyable if `T` is trivially copyable. If, first, either or both of the values of `x` or `w` of the **`operator()`** specified in this section is a NaN (Not a Number), ∞ or $-\infty$, secondly, NaN, ∞ or $-\infty$ occurs, or, thirdly, overflow or underflow occurs, which might even occur in the case of finite ranges of values, the function `value` returns an unspecified value.

5.2.1 Mean Accumulator Class Templates

```
template<class T>
class mean_accumulator
{
public:
  constexpr mean_accumulator() noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_mean_accumulator
{
public:
  constexpr weighted_mean_accumulator() noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr mean_accumulator() noexcept;
constexpr weighted_mean_accumulator() noexcept;
```

1. *Effects*: A (weighted) mean accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 1 invocation of **operator**().
2. *Effects*: Any remaining computations relating to the (weighted) mean are performed.
3. *Returns*: The (weighted) mean of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.2 Geometric Mean Accumulator Class Templates

```
template<class T>
class geometric_mean_accumulator
{
public:
  constexpr geometric_mean_accumulator() noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_geometric_mean_accumulator
{
public:
  constexpr weighted_geometric_mean_accumulator() noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr geometric_mean_accumulator() noexcept;
constexpr weighted_geometric_mean_accumulator() noexcept;
```

1. *Effects*: A (weighted) geometric mean accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 1 invocation of **operator**() and, if the product of the values `x` is negative, then the number of invocations of **operator**() is odd.
2. *Effects*: Any remaining computations relating to the (weighted) geometric mean are performed.
3. *Returns*: The (weighted) geometric mean of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.3 Harmonic Mean Accumulator Class Templates

```
template<class T>
class harmonic_mean_accumulator
{
public:
  constexpr harmonic_mean_accumulator() noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_harmonic_mean_accumulator
{
public:
  constexpr weighted_harmonic_mean_accumulator() noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr harmonic_mean_accumulator() noexcept;
constexpr weighted_harmonic_mean_accumulator() noexcept;
```

1. *Effects*: A (weighted) harmonic mean accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 1 invocation of **`operator`**`()` and and all of the values `x` are positive.
2. *Effects*: Any remaining computations relating to the (weighted) harmonic mean are performed.
3. *Returns*: The (weighted) harmonic mean of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.4 Variance Accumulator Class Templates

```
template<class T>
class variance_accumulator
{
public:
  constexpr variance_accumulator(T ddof) noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
```

```
class weighted_variance_accumulator
{
public:
  constexpr weighted_variance_accumulator(std::statistics_data_kind dkind) noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr variance_accumulator(T ddof) noexcept;
constexpr weighted_variance_accumulator(std::statistics_data_kind dkind) noexcept;
```

1. *Effects*: A (weighted) variance accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 1 invocation of **`operator`**`()` and `ddof` is not equal to the number of invocations of **`operator`**`()`.
2. *Effects*: Any remaining computations relating to the (weighted) variance are performed.
3. *Returns*: The (weighted) variance of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.5 Standard Deviation Accumulator Class Templates

```
template<class T>
class standard_deviation_accumulator
{
public:
  constexpr standard_deviation_accumulator(T ddof) noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_standard_deviation_accumulator
{
public:
  constexpr weighted_standard_deviation_accumulator(
    std::statistics_data_kind dkind) noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr standard_deviation_accumulator(T ddof) noexcept;
constexpr weighted_standard_deviation_accumulator(
  std::statistics_data_kind dkind) noexcept;
```

1. *Effects*: A (weighted) standard deviation accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 1 invocation of **`operator()`** and `ddof` is not equal to the number of invocations of **`operator()`**.
2. *Effects*: Any remaining computations relating to the (weighted) standard deviation are performed.
3. *Returns*: The (weighted) standard deviation of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.6 Skewness Accumulator Class Templates

```
template<class T>
class skewness_accumulator
{
public:
  constexpr skewness_accumulator(
    std::statistics_data_kind dkind, std::statistics_skewness_kind skind) noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_skewness_accumulator
{
public:
  constexpr weighted_skewness_accumulator(
    std::statistics_data_kind dkind, std::statistics_skewness_kind skind) noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr skewness_accumulator(
  std::statistics_data_kind dkind, std::statistics_skewness_kind skind) noexcept;
constexpr weighted_skewness_accumulator(
  std::statistics_data_kind dkind, std::statistics_skewness_kind skind) noexcept;
```

1. *Effects*: A (weighted) skewness accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.

2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 3 invocations of `operator()`.
2. *Effects*: Any remaining computations relating to the (weighted) skewness are performed.
3. *Returns*: The (weighted) skewness of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.7 Kurtosis Accumulator Class Templates

```
template<class T>
class kurtosis_accumulator
{
public:
  constexpr kurtosis_accumulator(
    std::statistics_data_kind dkind, std::statistics_kurtosis_kind kkind) noexcept;
  constexpr void operator()(const T& x);
  constexpr T value();
};

template<class T, class W = T>
class weighted_kurtosis_accumulator
{
public:
  constexpr weighted_kurtosis_accumulator(
    std::statistics_data_kind dkind, std::statistics_kurtosis_kind kkind) noexcept;
  constexpr void operator()(const T& x, const W& w);
  constexpr T value();
};
```

```
constexpr kurtosis_accumulator(
  std::statistics_data_kind dkind, std::statistics_kurtosis_kind kkind) noexcept;
constexpr weighted_kurtosis_accumulator(
  std::statistics_data_kind dkind, std::statistics_kurtosis_kind kkind) noexcept;
```

1. *Effects*: A (weighted) kurtosis accumulator object is constructed.
2. *Complexity*: Constant.

```
constexpr void operator()(const T& x);
constexpr void operator()(const T& x, const W& w);
```

1. *Effects*: The value of `x` (weighted by `w`) is accumulated.
2. *Complexity*: Constant.

```
constexpr T value();
```

1. *Preconditions*: At least 4 invocations of `operator()`.
2. *Effects*: Any remaining computations relating to the (weighted) kurtosis are performed.
3. *Returns*: The (weighted) kurtosis of the values of `x` (weighted by the values `w`) if the preconditions have been met and an unspecified value otherwise.
4. *Complexity*: Constant.

5.2.8 Accumulator Objects Accumulation Functions

```
template<ranges::input_range R, class ...Accumulators>
constexpr void statistics_accumulate(R&& r, Accumulators&&... acc);

template<ranges::input_range R, ranges::input_range W, class ...Accumulators>
constexpr void statistics_accumulate(R&& r, W&& w, Accumulators&&... acc);

template<class ExecutionPolicy, ranges::input_range R, class ...Accumulators>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
void statistics_accumulate(ExecutionPolicy&& policy, R&& r, Accumulators&&... acc);

template<class ExecutionPolicy,
  ranges::input_range R, ranges::input_range W,
  class ...Accumulators>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
void statistics_accumulate(ExecutionPolicy&& policy, R&& r, W&& w, Accumulators&&... acc);
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where the length of `r` is less than or equal to the length of `w`, and `acc` are valid accumulator objects.
2. *Effects*: The (weighted) statistics of the accumulator objects `acc` of the values of `r` (weighted by the corresponding values of `w`) are computed.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3 Freestanding Functions

If, first, either or both of the values of the ranges `r` or `w` of the functions specified in this section is a NaN, ∞ or $-\infty$, secondly, NaN, ∞ or $-\infty$ occurs, or, thirdly, overflow or underflow occurs, which might even occur in the case of finite ranges of values, the function returns an unspecified value.

5.3.1 Freestanding Mean Functions

```
template<ranges::input_range R>
constexpr auto mean(R&& r) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto mean(ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 1 value and the length of `r` is less than or equal to the length of `w`.
2. *Returns*: The (weighted) mean of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.2 Freestanding Geometric Mean Functions

```
template<ranges::input_range R>
constexpr auto geometric_mean(R&& r) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto geometric_mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto geometric_mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto geometric_mean(
  ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 1 value and the length of `r` is less than or equal to the length of `w`, and, if the product of the values of `r` is negative, then `ranges::distance(r)` is odd.
2. *Returns*: The (weighted) geometric mean of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.3 Freestanding Harmonic Mean Functions

```
template<ranges::input_range R>
constexpr auto harmonic_mean(R&& r) -> ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto harmonic_mean(R&& r, W&& w) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto harmonic_mean(ExecutionPolicy&& policy, R&& r) -> ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto harmonic_mean(
  ExecutionPolicy&& policy, R&& r, W&& w) -> ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 1 value and the length of `r` is less than or equal to the length of `w`, and all of the values of `r` are positive.
2. *Returns*: The (weighted) harmonic mean of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.4 Freestanding Variance Functions

```
template<ranges::input_range R>
constexpr auto variance(R&& r, typename ranges::iterator_t<R>::value_type ddof) ->
  ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto variance(R&& r, W&& w, std::statistics_data_kind dkind) ->
```

```
  ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto variance(
  ExecutionPolicy&& policy,
  R&& r,
  typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto variance(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  std::statistics_data_kind dkind) ->
    ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 1 value and the length of `r` is less than or equal to the length of `w`, and `ddof` is not equal to `ranges::distance(r)`.
2. *Returns*: The (weighted) variance of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.5 Freestanding Standard Deviation Functions

```
template<ranges::input_range R>
constexpr auto standard_deviation(
  R&& r, typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto standard_deviation(R&& r, W&& w, std::statistics_data_kind dkind) ->
  ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto standard_deviation(
  ExecutionPolicy&& policy,
  R&& r,
  typename ranges::iterator_t<R>::value_type ddof) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
auto standard_deviation(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  std::statistics_data_kind dkind) ->
    ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 1 value and the length of `r` is less than or equal to the length of `w`, and `ddof` is not equal to `ranges::distance(r)`.
2. *Returns*: The (weighted) standard deviation of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.6 Freestanding Skewness Functions

```
template<ranges::input_range R>
constexpr auto skewness(
  R&& r, statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto skewness(
  R&& r, W&& w, statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto skewness(
  ExecutionPolicy&& policy,
  R&& r,
  statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto skewness(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  statistics_data_kind dkind, statistics_skewness_kind skind) ->
    ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 3 values and the length of `r` is less than or equal to the length of `w`.
2. *Returns*: The (weighted) skewness of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

5.3.7 Freestanding Kurtosis Functions

```
template<ranges::input_range R>
constexpr auto kurtosis(
  R&& r, statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<ranges::input_range R, ranges::input_range W>
constexpr auto kurtosis(
  R&& r, W&& w, statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
constexpr auto kurtosis(
  ExecutionPolicy&& policy,
  R&& r,
  statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;

template<class ExecutionPolicy, ranges::input_range R, ranges::input_range W>
requires std::is_execution_policy_v<std::remove_cvref_t<ExecutionPolicy>>
```

```
constexpr auto kurtosis(
  ExecutionPolicy&& policy,
  R&& r, W&& w,
  statistics_data_kind dkind, statistics_kurtosis_kind kkind) ->
    ranges::iterator_t<R>::value_type;
```

1. *Preconditions*: `r` and `w` are ranges of finite values, where `r` has at least 4 values and the length of `r` is less than or equal to the length of `w`.
2. *Returns*: The (weighted) kurtosis of the values of `r` (weighted by the corresponding values of `w`) if the preconditions have been met and an unspecified value otherwise.
3. *Complexity*: Linear in `ranges::distance(r)`.

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Appendix A Examples

The following example showcases the use of **mean**, **variance** and **standard deviation** functions.

```
struct PRODUCT {
  float price;
  int quantity;
};

std::array<PRODUCT, 5> A = { {{5.2f, 1}, {1.7f, 2}, {9.2f, 5}, {4.4f, 7}, {1.7f, 3}} };
```

```
auto A_ = A
  | std::views::transform([](const auto& product) { return product.price; })
  | std::ranges::to<std::vector<float>>();
auto W = std::array<float, 5>{ 0.2f, 0.2f, 0.1f, 0.25f, 0.25f };

std::cout << "mean = " << std::mean(std::execution::par, A_);
std::cout << "\nvariance = " << std::variance(A_, 0);
std::cout << "\nstandard deviation = " << std::standard_deviation(
  A_, W, std::statistics_data_kind::population);
```

The following example showcases the use of a **kurtosis** function.

```
std::vector<double> v = { 2.0, 3.0, 5.0, 7.0, 11.0, 13.0, 17.0, 19.0 };
std::vector<double> v_wgts = { 0.2, 0.1, 0.3, 0.05, 0.05, 0.05, 0.1, 0.15 };

std::cout << "kurtosis = " << std::kurtosis(v, v_wgts,
  std::statistics_data_kind::population, std::statistics_kurtosis_kind::excess);
```

The following example showcases the use of **mean** accumulator objects.

```
std::mean_accumulator<int> m;
std::geometric_mean_accumulator<int> gm;
std::harmonic_mean_accumulator<int> hm;

std::statistics_accumulate(std::list<int>{ 3, 3, 1, 2, 2, 9 }, m, gm, hm);

std::cout << "mean = " << m.value();
std::cout << "\ngeometric mean = " << gm.value();
std::cout << "\nharmonic mean = " << hm.value();
```

The following example showcases the use of **mean**, **skewness** and **custom** accumulator objects.

```
/* custom accumulator */
class sum_squares_accumulator
{
public:
  constexpr sum_squares_accumulator() noexcept { sum_squares_=0; }
  constexpr void operator()(double x) { sum_squares_ += x*x; }
  constexpr double value() { return sum_squares_; }
private:
  double sum_squares_;
};

// ...

std::list<int> L = { 3, 3, 1, 2, 2, 9 };

std::mean_accumulator<int> m;
std::skewness_accumulator<int> sk(
  std::statistics_data_kind::population, std::statistics_skewness_kind::unadjusted);
sum_squares_accumulator ssq;

std::statistics_accumulate(L, m, sk, ssq);

std::cout << "mean = " << m.value();
std::cout << "\nskewness = " << sk.value();
std::cout << "\nsum of squares = " << ssq.value();
```

Appendix B Derivations of Skewness and Kurtosis Formulas

Building on the results of [21], derivations of the (less familiar) skewness and kurtosis formulas of Sections 2.4 and 2.5, respectively, are presented in what follows. Let m_2, m_3 and m_4 be the *second*, *third* and *fourth* **population** *central moments* [21, 43], respectively, of the values, defined as

$$m_2 = \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu\right)^2, \tag{24}$$

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu\right)^3 \text{ and} \tag{25}$$

$$m_4 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \mu\right)^4. \tag{26}$$

Let M_2, M_3 and M_4 be the second, third and fourth **sample** central moments [21], respectively, defined as

$$M_2 = \frac{n}{n-1}m_2, \tag{27}$$

$$M_3 = \frac{n^2}{(n-1)(n-2)}m_3 \text{ and} \tag{28}$$

$$M_4 = \frac{n\left(n^2-2n+3\right)}{(n-1)(n-2)(n-3)}m_4 - \frac{3n(2n-3)}{(n-1)(n-2)(n-3)}m_2^2. \tag{29}$$

Let t_1, t_2 and t_3 be the **weighted** second, third and fourth **population** central moments [21], respectively, defined as

$$t_2 = \hat{\sigma}^2 = \frac{1}{V_1}\sum_{i=1}^{n}w_i\left(x_i - \mu^*\right)^2, \tag{30}$$

$$t_3 = \frac{1}{V_1}\sum_{i=1}^{n}w_i\left(x_i - \mu^*\right)^3 \text{ and} \tag{31}$$

$$t_4 = \frac{1}{V_1}\sum_{i=1}^{n}w_i\left(x_i - \mu^*\right)^4. \tag{32}$$

Let T_2, T_3 and T_4 be the **weighted** second, third and fourth **sample** central moments [21], respectively, defined as

$$T_2 = \frac{V_1^2}{V_1^2 - V_2}t_2, \tag{33}$$

$$T_3 = \frac{V_1^3}{V_1^3 - 3V_1V_2 + 2V_3}t_3 \text{ and} \tag{34}$$

$$T_4 = \frac{V_1^2\left(V_1^4 - 3V_1^2V_2 + 2V_1V_3 + 3V_2^2 - 3V_4\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}t_4 - \frac{3V_1^2\left(2V_1^2V_2 - 2V_1V_3 - 3V_2^2 + 3V_4\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}t_2^2. \tag{35}$$

Let c_4 be the *fourth* **population** *cumulant* [21, 44], defined as

$$c_4 = m_4 - 3m_2^2. \tag{36}$$

Let C_4 be the *fourth* **sample** *cumulant* [21], defined as

$$C_4 = \frac{n^2(n+1)}{(n-1)(n-2)(n-3)}m_4 - \frac{3n^2}{(n-2)(n-3)}m_2^2. \tag{37}$$

Let k_4 be the **weighted** *fourth* **population** *cumulant* [21], defined as

$$k_4 = t_4 - 3t_2^2. \tag{38}$$

Let K_4 be the **weighted** *fourth* **sample** *cumulant* [21], defined as

$$K_4 = \frac{V_1^2\left(V_1^4 - 4V_1V_3 + 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}t_4 - \frac{3V_1^2\left(V_1^4 - 2V_1^2V_2 + 4V_1V_3 - 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}t_2^2. \tag{39}$$

B.1 Skewness

The **population** skewness [21] is

$$\frac{m_3}{m_2^{3/2}} = \frac{m_3}{\sigma^3} \tag{40}$$

$$= \frac{1}{n\sigma^3}\sum_{i=1}^{n}\left(x-\mu\right)^3. \tag{41}$$

The **sample** skewness [21] is

$$\frac{M_3}{M_2^{3/2}} = \frac{\dfrac{n^2}{\left(n-1\right)\left(n-2\right)}m_3}{\left(\dfrac{n}{n-1}m_2\right)^{3/2}} \tag{42}$$

$$= \frac{\dfrac{n^2}{\left(n-1\right)\left(n-2\right)}m_3}{\left(s^2\right)^{3/2}} \tag{43}$$

$$= \frac{n}{\left(n-1\right)\left(n-2\right)s^3}\sum_{i=1}^{n}\left(x-\bar{x}\right)^3. \tag{44}$$

The **weighted population** skewness [21] is

$$\frac{t_3}{t_2^{3/2}} = \frac{t_3}{\hat{\sigma}^3} \tag{45}$$

$$= \frac{1}{V_1\sigma^3}\sum_{i=1}^{n}w_i\left(x-\mu^*\right)^3. \tag{46}$$

The **weighted sample** skewness [21] is

$$\frac{T_3}{T_2^{3/2}} = \frac{\dfrac{V_1^3}{V_1^3-3V_1V_2+2V_3}t_3}{\left(\dfrac{V_1^2}{V_1^2-V_2}t_2\right)^{3/2}} \tag{47}$$

$$= \frac{V_1^3}{V_1^3-3V_1V_2+2V_3}t_3\cdot\frac{\left(V_1^2-V_2\right)^{3/2}}{V_1^3\left(t_2\right)^{3/2}} \tag{48}$$

$$= \frac{\left(V_1^2-V_2\right)^{3/2}}{V_1\left(V_1^3-3V_1V_2+2V_3\right)\hat{\sigma}^3}\sum_{i=1}^{n}w_i\left(x-\bar{x}^*\right)^3. \tag{49}$$

B.2 Kurtosis

The Pearson **population** kurtosis [21, 45] is

$$\frac{m_4}{m_2^2} = \frac{m_4}{\sigma^4} \tag{50}$$

$$= \frac{1}{n\sigma^4}\sum_{i=1}^{n}\left(x-\mu\right)^4. \tag{51}$$

The Pearson **sample** kurtosis [21] is

$$\frac{M_4}{M_2^2} = \frac{\frac{n\left(n^2-2n+3\right)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n(2n-3)}{(n-1)(n-2)(n-3)} m_2^2}{\left(\frac{n}{n-1} m_2\right)^2} \tag{52}$$

$$= \frac{\frac{n\left(n^2-2n+3\right)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n(2n-3)}{(n-1)(n-2)(n-3)} \sigma^4}{\left(s^2\right)^2} \tag{53}$$

$$= \frac{n^2-2n+3}{(n-1)(n-2)(n-3)s^4} \sum_{i=1}^{n} (x_i-\bar{x})^4 - \frac{3n(2n-3)\sigma^4}{(n-1)(n-2)(n-3)s^4}. \tag{54}$$

The **weighted** Pearson **population** kurtosis [21] is

$$\frac{t_4}{t_2^2} = \frac{t_4}{\hat{\sigma}^4} \tag{55}$$

$$= \frac{1}{n\hat{\sigma}^4} \sum_{i=1}^{n} (x-\mu^*)^4. \tag{56}$$

The **weighted** Pearson **sample** kurtosis [21] is

$$\frac{T_4}{T_2^2} = \frac{\frac{V_1^2\left(V_1^4-3V_1^2V_2+2V_1V_3+3V_2^2-3V_4\right)}{\left(V_1^2-V_2\right)\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)} t_4 - \frac{3V_1^2\left(2V_1^2V_2-2V_1V_3-3V_2^2+3V_4\right)}{\left(V_1^2-V_2\right)\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)} t_2^2}{\left(\frac{V_1^2}{V_1^2-V_2} t_2\right)^2} \tag{57}$$

$$= \frac{V_1^2\left(V_1^4-3V_1^2V_2+2V_1V_3+3V_2^2-3V_4\right)}{\left(V_1^2-V_2\right)\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)} t_4 - \tag{58}$$

$$\frac{3V_1^2\left(2V_1^2V_2-2V_1V_3-3V_2^2+3V_4\right)}{\left(V_1^2-V_2\right)\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)} t_2^2 \cdot \frac{\left(V_1^2-V_2\right)^2}{V_1^4\left(t_2\right)^2} \tag{59}$$

$$= \frac{\left(V_1^2-V_2\right)\left(V_1^4-3V_1^2V_2+2V_1V_3+3V_2^2-3V_4\right)}{V_1^3\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)\hat{\sigma}^4} \sum_{i=1}^{n} w_i(x_i-\bar{x}^*)^4 - \frac{3\left(V_1^2-V_2\right)\left(2V_1^2V_2-2V_1V_3-3V_2^2+3V_4\right)}{V_1^2\left(V_1^4-6V_1^2V_2+8V_1V_3+3V_2^2-6V_4\right)}. \tag{60}$$

The excess **population** kurtosis [21] is

$$\frac{c_4}{m_2^2} = \frac{m_4-3m_2^2}{m_2^2} \tag{61}$$

$$= \frac{m_4}{\sigma^4} - 3 \tag{62}$$

$$= \frac{1}{n\sigma^4} \sum_{i=1}^{n} (x-\mu)^4 - 3. \tag{63}$$

The excess **sample** kurtosis [21] is

$$\frac{C_4}{M_2^2} = \frac{\frac{n^2(n+1)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n^2}{(n-2)(n-3)} m_2^2}{\left(\frac{n}{n-1} m_2\right)^2} \tag{64}$$

$$= \frac{\frac{n^2(n+1)}{(n-1)(n-2)(n-3)} m_4 - \frac{3n^2}{(n-2)(n-3)} m_2^2}{\left(s^2\right)^2} \tag{65}$$

$$= \frac{n(n+1)}{(n-1)(n-2)(n-3)s^4} \sum_{i=1}^{n} (x_i-\bar{x})^4 - \frac{3n^2\sigma^4}{(n-2)(n-3)s^4}. \tag{66}$$

The **weighted** excess **population** kurtosis [21] is

$$\frac{k_4}{t_2^2} = \frac{t_4 - 3t_2^2}{t_2^2} \tag{67}$$

$$= \frac{t_4}{\hat{\sigma}^4} - 3 \tag{68}$$

$$= \frac{1}{n\hat{\sigma}^4} \sum_{i=1}^{n} (x - \mu^*)^4 - 3. \tag{69}$$

The **weighted** excess **sample** kurtosis [21] is

$$\frac{K_4}{T_2^2} = \frac{\frac{V_1^2\left(V_1^4 - 4V_1V_3 + 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)} t_4 - \frac{3V_1^2\left(V_1^4 - 2V_1^2V_2 + 4V_1V_3 - 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)} t_2^2}{\left(\frac{V_1^2}{V_1^2 - V_2} t_2\right)^2} \tag{70}$$

$$= \frac{V_1^2\left(V_1^4 - 4V_1V_3 + 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)} t_4 - \tag{71}$$

$$\frac{3V_1^2\left(V_1^4 - 2V_1^2V_2 + 4V_1V_3 - 3V_2^2\right)}{\left(V_1^2 - V_2\right)\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)} t_2^2 \cdot \frac{\left(V_1^2 - V_2\right)^2}{V_1^4 \left(t_2\right)^2} \tag{72}$$

$$= \frac{\left(V_1^2 - V_2\right)\left(V_1^4 - 4V_1V_3 + 3V_2^2\right)}{V_1^3\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)\hat{\sigma}^4} \sum_{i=1}^{n} w_i(x_i - \bar{x}^*)^4 - \frac{3\left(V_1^2 - V_2\right)\left(V_1^4 - 2V_1^2V_2 + 4V_1V_3 - 3V_2^2\right)}{V_1^2\left(V_1^4 - 6V_1^2V_2 + 8V_1V_3 + 3V_2^2 - 6V_4\right)}. \tag{73}$$